Optimal intervention in a random-matching model of money*

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July 5, 2016

Abstract

Wallace [2014] conjectures that there generically exists an inflation-financed transfer scheme that improves welfare over no intervention in pure-currency economies. We investigate this conjecture in the Shi-Trejos-Wright model with different upper bounds on money holdings. The choice of upper bound affects the results as some potentially beneficial transfer schemes cannot be studied under small upper bounds. Numerical optima are computed for different degrees of discounting rate and risk aversion. As the upper bound on money holdings increases, optima are more likely to have positive money creation (and inflation), and this is in line with the conjecture.

1 Introduction

Wallace [2014] conjectures that in a class of economies in which all trade must involve money and there is no explicit taxation, there exist beneficial inflation-financed transfer schemes (intervention). A simple class of such schemes that he discusses involves a transfer to a person with m amount of money equal to $\max\{0,\ a + bm\}$, where $b \geq 0$.[1] The conjecture applies to economies in which trades and policies affect both current-period payoffs and future states of the economy, the typical situation in heterogeneous-agent economies. Wallace discusses two

*We are grateful to Neil Wallace for his valuable comments and support. We thank Makoto Watanabe for helpful comments.

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1When $a = 0$, such transfers are neutral; when $a > 0$, they are equivalent to lump-sum transfers; and when $a < 0$, they pay interest on money to those with holdings that exceed $-a/b$.

examples, the alternating endowment economy with random "switches" (see Levine [1991] and Kehoe et al. [1992]) and a random-matching model due to Shi [1995] and Trejos and Wright [1995], but with a rich set of individual money holdings. However, he presents results only for the former. Here, we present numerical results for a version of the latter model.

In order to do that, we are forced to study a version with a relatively small set of individual money holdings, $\{1, 2, \ldots, B\}$ with B small. Both the discreteness and the bound force us to adapt the policies and the way we model the inflation that results from the transfers. The bound limits transfers to those at the bound. The discreteness forces both transfers and inflation to be probabilistic, where inflation is modeled as a probabilistic version of a proportional tax on money holdings—a tax which is nothing but a normalization when money is divisible. Our main results are for $B = 3$, which, as described below, is mainly dictated by computational feasibility. This magnitude of B is interesting because it is the smallest B that gives potential scope to policies with $a > 0$ and to policies with $a < 0$. Deviatov [2006] studies optima under $B = 2$ and we repeat that case here, but that case does not give scope to policies with $a < 0$. Policies with $a < 0$ require a positive holding with transfers given only to those with higher holdings. That positive holding must be at least one. With $B = 2$, all holdings that exceed one are at the bound and, therefore, ineligible for a transfer. With $B = 3$, those with two units of money can receive a transfer, a transfer which gives an additional incentive to those with one unit to produce and acquire additional money.

As in Deviatov [2006], we study alternative steady states in which the planner is choosing the steady-state distribution of money holdings, the trades in meetings subject to those trades being in the pairwise core in each meeting, and the above policies in order to maximize ex ante representative-agent utility. We present results for various combinations of two aspects of preferences: the discount factor and the finite marginal utility of consumption at zero. Consistent with Deviatov [2006], when $B = 2$ there are few cases in which intervention - inflation-financed government transfer - helps and those interventions have $a > 0$, are lump-sum transfers. When $B = 3$, more cases have desirable intervention; some have $a > 0$ and others have $a < 0$. We also made attempts to study optima for $B = 4$, but we could not get reliable results for all the parameter combinations. Nevertheless, the findings are broadly consistent with the surmise that the set of parameters for which no-intervention is optimal shrinks as B gets larger.

2 Environment

The environment is borrowed from a random matching model in Shi [1995] and Trejos and Wright [1995]. Time is discrete and the horizon is infinite. There are a nonatomic measure of infinitely-lived agents. In each period, pairwise meetings for production and consumption occur in the following way. An agent becomes a producer (who meets a random consumer) with probability $\frac{1}{K}$, becomes a consumer (who meets a random producer) with probability $\frac{1}{K}$, or becomes inactive and enters no meeting with probability $1-\frac{2}{K}$. In a meeting, the producer can produce q units of a consumption good for the consumer in the meeting at the cost of disutility $c(q)$, where c is strictly increasing, convex, and differentiable and $c(0)=0$. The consumer obtains period utility $u(q)$, where u is strictly increasing, strictly concave, differentiable function on $\mathbb{R}_+$ and satisfies $u(0)=0$. The consumption good is perishable: it must be consumed in a meeting or discarded. Agents maximize the expected sum of discounted period utilities with discount factor $\beta \in (0,1)$.

Individual money holdings are restricted to be in $\{0,1,...,B\}$. The state of the economy entering a date is a distribution over that set. Then there are pairwise meetings at random at which lottery trades occur: in single coincidence meetings some amount of output goes from the producer to the consumer and there is a lottery that determines the amount of money that the consumer gives the producer. Next, there are transfers. We let $\tau_i \geq 0$ be the transfer to a person who ends trade with i units and impose only that τ_i is weakly increasing in i for $i \in \{0,1,...,B-1\}$ and that $\tau_B = 0$. Finally, inflation occurs via probabilistic distintegration of money. Each unit of money held disappears with probability δ.

We assume that people cannot commit to future actions and that there is no public monitoring in the sense that histories of agents are private. However, we assume that money holdings and consumer-producer status are known within meetings, but that money holdings are private at the transfer stage which is why we assume that τ_i is weakly increasing in i for $i \in \{0,1,...,B-1\}$.

All our computations are for $K=3$, $c(q)=q$, and $u(q)=1-e^{-\kappa q}$, which implies that $u'(0)=\kappa$. We study optima for a subset of

$$(\beta,\kappa) \in \{0.15,0.2,0.25,0.3,0.35,0.4,0.5,0.6,0.7,0.8\} \times \{2,3,4,5,6,8,10,12,15,20\},$$

a subset that satisfies

$$\kappa > 1 + \frac{K(1-\beta)}{\beta}. \tag{1}$$

This condition is necessary and sufficient for the production of constant positive output in a version of the model with perfect monitoring.[2] It is necessary for existence of a monetary equilibrium. When it is imposed the subset contains 55 elements.

3 The planner's problem

We study allocations that are stationary and symmetric meaning that agents in the same situation (money holdings, producer-consumer status) take the same action. Therefore, productions and monetary payments are constant over all meetings in which a producer has k units of money and a consumer has k' units of money, a (k, k') meeting. A stationary and symmetric allocation consists of the variables listed in Table 1:

Table 1: Variables constituting an allocation

π_k	fraction with k units of money before meetings
$q(k, k')$	production in (k, k') meeting
$\lambda_p^{k,k'}(i)$	probability that producer has i money after (k, k') meeting
$\lambda_c^{k,k'}(i)$	probability that consumer has i money after (k, k') meeting
τ_k	transfer rate for agents with k units of money
δ	probability that money disintegrates after meetings

The planner chooses production and payment in every meeting, disintegration and transfer rates to maximize ex-ante expected utility before money are assigned according to the stationary distribution. It can be easily shown that ex-ante expected utility is proportional to the expected gains from trade in meetings:

$$\sum_{0 \le k \le B} \sum_{0 \le k' \le B} \pi_k \pi_{k'} \left[u(q(k, k')) - q(k, k') \right] \tag{2}$$

The planner is subject to the following constraints.

Physical feasibility and stationarity First, money holdings resulting from meetings must be feasible within the pair: in (k, k') meeting, if the consumer has i units, then the producer must have $k + k' - i$ units. Also, money holdings cannot be negative or exceed the total amount brought into the meeting.

[2]The inequality is derived from

$$\frac{d}{dq}_{q=0} \left(-c(q) + \frac{\beta}{K(1-\beta)} \left[u(q) - c(q) \right] \right) > 0$$

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(k+k'-i) \text{ if } 0 \le i \le k+k' \tag{3}$$

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(i) = 0 \text{ if } i < 0 \text{ or } k+k' < i \tag{4}$$

Let $\Lambda(k,k')$ denote the set of pairs of probabilities $(\lambda_c^{k,k'}, \lambda_p^{k,k'})$ that satisfy the above constraints.

The money holding distribution is required to be stationary and consistent with transition probability specified by monetary payments in meetings, disintegration and transfer rates. Given a money holding distribution $\{\pi_k\}_{k\in\{0,1,\dots,B\}}$ and a money holding transition probability for each meeting $\{\lambda_p^{k,k'}(i), \lambda_c^{k,k'}(i)\}_{(k,k',i)\in\{0,1,\dots,B\}^3}$, the transition probability that an agent with k units of money before meeting ends up with k' units of money after meeting is

$$t^{(1)}(k,k') = \frac{1}{K}\sum_{i\in\{0,\dots,B\}} \pi_i \left[\lambda_p^{k,i}(k') + \lambda_c^{i,k}(k')\right] + \frac{K-2}{K} 1_{k=k'}.$$

The transition caused by a transfer is expressed by

$$t^{(2)}(k,k') = \begin{cases} 1-\tau_k & \text{if } k < B \text{ and } k' = k, \\ \tau_k & \text{if } k < B \text{ and } k' = k+1, \\ 1 & \text{if } k' = k = B, \\ 0 & \text{otherwise,} \end{cases}$$

and the transition caused by disintegration is expressed by

$$t^{(3)}(k,k') = \begin{cases} \begin{pmatrix} k \\ k' \end{pmatrix} \delta^{k-k'}(1-\delta)^{k'} & \text{if } k \ge k', \\ 0 & \text{otherwise.} \end{cases}$$

Denote $T^{(i)}$ the matrix whose (n,n') elements are $t^{(i)}(n-1,n'-1)$. Specifically,

$$T^{(i)} \equiv \begin{bmatrix} t^{(i)}(0,0) & t^{(i)}(0,1) & \dots \\ t^{(i)}(1,0) & t^{(i)}(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The stationarity constraint can be stated as

$$\pi = \pi T^{(1)} T^{(2)} T^{(3)}, \tag{5}$$

where

$$\pi = \begin{bmatrix} \pi_0 & \cdots & \pi_B \end{bmatrix}.$$

Incentive compatibility We assume that agents can deviate individually and also cooperatively from trades, chosen by the planner, in the meeting stage. Individual deviations lead to no trade, and profitable cooperative deviations lead to a Pareto-improving alternative trade. To state incentive compatibility constraints arising from such deviations, it is convenient to use discounted expected utility. Discounted utility for an agent with k units of money before pairwise meeting is denoted by $v(k)$, and that after the meeting stage but before transfer and disintegration is denoted by $w(k)$. They are defined for each symmetric and stationary allocation in a standard way. Specifically, for each $k \in \{0, 1, ..., B\}$,

$$\begin{aligned} v(k) =& \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[u(q(k', k)) + \beta \sum_{0 \le i \le k+k'} \lambda_c^{k',k}(i) w(i) \right] \\ &+ \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[-q(k, k') + \beta \sum_{0 \le i \le k+k'} \lambda_p^{k,k'}(i) w(i) \right] \\ &+ \frac{K-2}{K} \beta w(k) \end{aligned} \tag{6}$$

$$w(k) = \sum_{k' \in \{0,\dots,B\}} t^{(2)}(k, k') \sum_{i \in \{0,\dots,B\}} t^{(3)}(k', i) v(i) \tag{7}$$

Trades are immune to both individual and cooperative deviations if post-trade allocations are in the pairwise core, and we call this pairwise core constraint. To state the constraint for (k, k') meeting, let $\vartheta(k, k')$ denote a surplus (over no-trade) for a producer in the meeting. The constraint can be stated as follows: $q(k, k')$, $\lambda_p^{k,k'}$, and $\lambda_c^{k,k'}$ solve

$$\begin{aligned} \max_{q \ge 0,\ (\lambda_p, \lambda_c) \in \Lambda(k,k')} & u(q) + \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w(i) \\ \text{s.t.} \quad & -q + \beta \sum_{0 \le i \le k+k'} \lambda_p(i) w(i) = \beta w(k) + \vartheta(k, k') \\ & u(q) + \beta \sum_{0 \le i \le k+k'} \lambda_c(i) w(i) \ge \beta w(k') \end{aligned} \tag{8}$$

for some $\vartheta(k, k') \ge 0$.[3] The Karush-Kuhn-Tucker condition is necessary and sufficient for

[3] Solving the problem is necessary for trades being in the pairwise core. That is also sufficient if the utility function of the producer and the consumer are strictly monotone in consumption goods and money holdings (see, for example, Mas-Collel et al. [1995]). Here, the utility function may not be strictly increasing in money holdings; Some additional units of money may not be valued in some allocations, and hence the value function w, which specifies the preference for money holdings in trade meetings, may be non-strictly

the optimality, and we can derive a set of equations and inequalities from the condition. See Appendix A for the detail.

The planner maximizes the ex-ante expected utility (2), subject to the physical feasibility conditions, the stationarity conditions, and the pairwise core constraints.

4 Computational procedure

We compute solutions for the planner's problem using two solvers that are compatible with the GAMS interface, KNITRO and BARON. KNITRO is a local solver for large scale optimization problems. For a given initial point, it quickly converges to a local solution (or shows that it cannot reach one), but it does not guarantee global optimality. This issue is usually dealt with by using a large number of initial values. The solver automatically feeds in different initial values as we change an option that controls the number of initial values. In contrast, BARON (Branch-And-Reduce Optimization Navigator) is a global solver for nonconvex optimization problems. It continues to update an upper bound and a lower bound on the objective by evaluating the values of variables satisfying the constraints, and stops when the difference between the two bounds becomes smaller than a threshold. It guarantees global optimality under mild conditions, but it generally takes much longer time to converge than local solvers. Even before it converges, we can terminate it and see its candidate solution. When Baron did not finish in a reasonable time span, we stopped it and checked the candidate solution with the solution from KNITRO.

For $B = 2$, BARON under its default criterion usually finished in about an hour, and it reproduces the solution that KNITRO finds using 250 different initial points. For $B = 3$, BARON did not finish in 200 hours. In all cases that we tried, the candidate solution was not updated after roughly 20 hours. (The remaining time was being used to verify that other feasible allocations are not better than the candidate solution.) We ran KNITRO with 1000 initial points and found that its solution coincides with the intermediate output from BARON, which is the best lower bound. Also, to check whether the computation is sensitive to the number of initial points we use, we ran KNITRO with 8000 initial points and made sure that the results are the same.

We also tried the same approach with $B = 4$, but we could not find robust results for some examples. For $B = 4$, the risk aversion parameter κ is varied over $\{2, 3, 4, 5, 6, 8, 10, 12, 15, 20\}$ and the discount factor parameter β is varied over $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. Optimum

increasing in a part of the domain. In effect, we are solving a relaxed problem using this formulation. For example, if we find that a numerical solution has non-strictly increasing w, it may not be an optimum as solving above problem is not a sufficient condition in that case. It is verified that numerical solutions have strictly increasing w, and thus it is assured that the solutions solve the problem of our interest.

is computed for each of 46 pairs of κ and β, under which the condition (1) is satisfied, out of 70 possible pairs, using (i) KNITRO with 8000 initial points, (ii) KNITRO with 16000 initial points, or (iii) BARON.

We found the numerical results for $B = 4$ are less reliable than those for $B = 2$ or 3: In roughly a third of the cases, either KNITRO gives different answers depending on the number of initial conditions or KNITRO and BARON give different answers.

5 Results

With $B = 2$, we find 2 parameter combinations, $(\kappa, \beta) = (15, 0.7)$ and $(20, 0.6)$, where intervention was optimal. The transfer scheme used in these cases were lump-sum in the sense that they satisfy $\tau_0 = \tau_1$, and the transfer rates were 2.8% and 1.4%. To measure the improvement over no intervention, we compute the amount of consumption that agents are willing to give up to have optimal inflation (and transfer) instead of no-intervention. In particular, we calculate z that satisfies

$$\sum_{k,k'} \pi_k^* \pi_{k'}^* \left[u\left(\frac{100 - z}{100} q^*(k, k') \right) - c(q^*(k, k')) \right] = \sum_{k,k'} \pi_k^0 \pi_{k'}^0 \left[u(q^0(k, k')) - c(q^0(k, k')) \right],$$

where q^* is the optimal production with intervention and q^0 is the optimal production with no-intervention. The welfare gain from intervention (z) is 0.08% for $(\kappa, \beta) = (15, 0.7)$ and 0.40% for $(\kappa, \beta) = (20, 0.6)$.

With $B = 3$, some intervention was optimal in 21 parameter combinations. Moreover, all interventions were either lump-sum, $\tau_0 = \tau_1 = \tau_2 > 0$, or $\tau_0 = \tau_1 = 0$ and $\tau_2 > 0$. In other words, they turned out to fit the class of transfer scheme discussed in the introduction. Because all the solutions have $\tau_0 = \tau_1$, it is convenient to display them in a table with two numbers, x/y , where x is the common magnitude of τ_0 and τ_1, and y is the magnitude of τ_2. Table 2 reports the optimal transfer rates in such a way, and the measure of welfare gain from intervention (z) is underneath them.

The type of optimal transfer is related to the value of β and κ. Among cases in which some intervention is optimal, the optimal transfer tends to be lump-sum when β and κ are both high, and non-lump-sum when either β or κ is low. To understand this result, it is helpful to spell out the benefits and the costs of the two transfer types. The common cost for any transfers is the accompanying inflation, as it is lowering the producer's incentive. The benefit of lump-sum transfer is the risk-sharing. If an agent without money becomes a consumer, he must forego the opportunity to consume and this is wasteful from a society's

Table 2: Transfer (%) and welfare gain from intervention (%) , $B = 3$

$\kappa\backslash\beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0/0 (0)	0/72.3 (3.52)	0/73.4 (4.92)	0/73.7 (1.87)	0/0 (0)	3.53/3.53 (0.68)	3.12/3.12 (2.04)	13.0/13.0 (6.50)	11.4/11.4 (2.42)	0/0 (0)
15	-	0 /0 (0)	0/65.4 (2.70)	0/66.8 (3.71)	0/67.1 (0.37)	0/0 (0)	0/0 (0)	5.31/5.31 (1.40)	2.8/2.8 (0.48)	0/0 (0)
12	-	-	0/0 (0)	0/59.6 (2.48)	0/61.0 (2.44)	0/0 (0)	0/0 (0)	2.1/2.1 (0.21)	0/0 (0)	0/0 (0)
10	-	-	-	0/0 (0)	0/54.6 (2.52)	0/55.7 (1.05)	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
8	-	-	-	-	0/0 (0)	0/47.2 (1.18)	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
6	-	-	-	-	-	0/0 (0)	0/38.4 (0.58)	0/0 (0)	0/0 (0)	0/0 (0)
5	-	-	-	-	-	-	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
4	-	-	-	-	-	-	-	0/0.6 (0.12)	0/0 (0)	0/0 (0)
3	-	-	-	-	-	-	-	-	0/0.9 (0.18)	0/0 (0)
2	-	-	-	-	-	-	-	-	-	0/0 (0)

point of view. The transfer is helpful in reducing such loss. However, as people get free money from transfer regardless of their money holdings, this type of transfer further lowers producers' incentive to earn money. Hence it tightens producers' participation constraint. In contrast, the benefit of non-lump-sum transfer used in this example, where transfer rate is strictly increasing in an interval, is that it enhances producers' incentive, particularly those who already owns some money. Such transfer can relax the participation constraint of producers who already own some money.

The pattern of optimal transfer fits with the explanation on the benefit and the cost of each transfer type: when people have more incentive to work for future consumption and averse risk more (that is, when β and κ are both high), the optimal transfer tends to provide risk-sharing. If people have less incentive to work for future consumption and do not averse risk much (that is, when β and κ are both low), the optimal transfer tends to enhance the incentive of producers. Although there are some examples where no-intervention is optimal with the upper bound of three, we believe that the region where no-intervention is optimal will shrink as the upper bound increases, comparing the optimal transfer rates for $B = 2$ and $B = 3$. The results we attained with the upper bound of four are broadly consistent with this conjecture.

The gain varies from 0.12 percentage point to 6.50 percentage point. The largest gain is attained when the discount factor and the risk aversion are relatively high, and the optimal transfer is lump-sum.

We report details of optima for two examples, $(\kappa, \beta) = (15, 0.7)$ and $(\kappa, \beta) = (15, 0.3)$, in Table 3 and 4. In the former, the optimal transfer is lump-sum, while it goes only to someone with 2 units in the latter. Table 3 shows the money holding distribution π, the transfer τ, and the production and payment in each (k, k') meeting for $(\kappa, \beta) = (15, 0.7)$ and Table 4 shows those for $(\kappa, \beta) = (15, 0.3)$. On the top of each table, the production and payment in meetings are shown in a matrix. Each entry is in the form $q/(\lambda)$, where q is output as a fraction of the first-best level and λ is the expected monetary payment from the consumer to the producer. A star attached to each entry indicates that the participation constraint for the producer is binding in the meeting.

First, let us compare the distribution. The distribution for $(\kappa, \beta) = (15, 0.7)$ is more concentrated around the center, 1 and 2, achieving more trade meetings than that for $(\kappa, \beta) = (15, 0.3)$. The share of agents with i units of money is decreasing in i in the example for $(\kappa, \beta) = (15, 0.7)$, while $\pi_2 < \pi_3$ holds in the example for $(\kappa, \beta) = (15, 0.3)$ as a result of the non-lump-sum transfer.

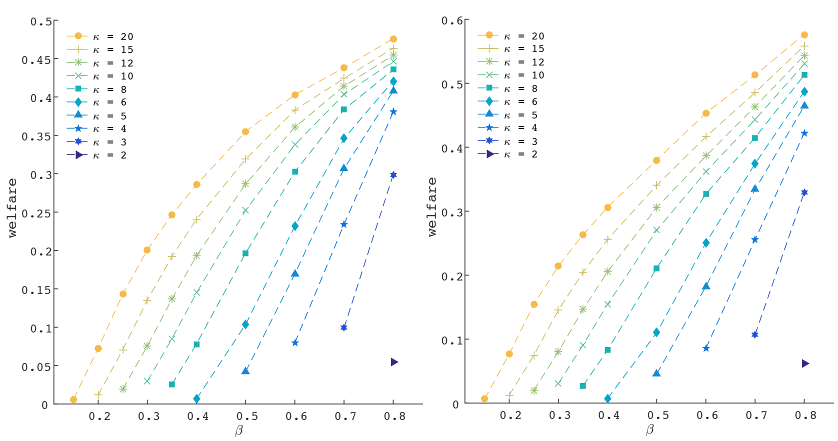
The production level is higher in the example for $(\kappa, \beta) = (15, 0.7)$ than in that for $(\kappa, \beta) = (15, 0.3)$ because of more patience. The monetary payment varies from 0 to 2 over meetings in $(\kappa, \beta) = (15, 0.3)$, while it is 1 in all meetings in $(\kappa, \beta) = (15, 0.7)$, achieving the concentrated money holding distribution mentioned above. This result is consistent with an observation in Deviatov [2006].

In the example for $(\kappa, \beta) = (15, 0.7)$, the Individual Rationality constraint for producer is not binding in the $(0, 1)$ and $(0, 2)$ meetings. That means, in these meetings, the terms of trade is not determined by take-it-or-leave-it offer by the consumer.We find some non-binding IR constraints in all examples in which lum-sum transfer is optimal, while we find all IR constraints are binding in all examples in which non-lump-sum transfer is optimal.

In principle, restricting trading protcol to take-it-or-leave-it offer by the consumer leaves more room for welfare improvements by intervention through transfer. Here, allowing any trades in pairwise core, we minimizes the necessity of welfare improvements through transfer. Our examples show that even in such a setting some intervention can be optimal.

We report some other features of optima in the following. , As expected, welfare at the optimum is increasing with discount factor (Figure 5). The welfare at optimum appears to increase with B as well. Related to this result, Zhu [2003] shows that the set of implementable allocations for lower n is a subset of the set for larger n when B increases as in $B_n = m^n$ for any integer $m > 1$. Hence, welfare is a weakly increasing function for n if B_n increases in that way. However, it is not straightforward whether the result will extend to $B_n = n$ for $n = 1, 2, \ldots$. At least to our knowledge, numerical results that compared welfare at the optimum for different B (greater than 1) do not exist. Our result is in line with the hypothesis of increasing welfare at optimum with respect to B.

Figure 1: Welfare at optimum with $B = 2,\ 3$



We define money supply as the average money holdings of people relative to the upper bound. Table 5 and 6 report the money supply for $B = 2$ and $B = 3$ respectively. Optimal money supply monotonically increases with discount factor and risk aversion. Comparing

Table 3: Optimal allocation for $(\kappa, \beta) = (15, 0.7)$

Distribution	π_0	0.346		
	π_1	0.283		
	π_2	0.216		
	π_3	0.155		
Transfer	τ_0	0.028		
	τ_1	0.028		
	τ_2	0.028		
production and payment in (k, k') meeting	$k \backslash k'$	1	2	3
	0	0.551/(1)	1.118/(1)	1.396/(1)*
	1	0.691/(1)*	0.691/(1)*	0.691/(1)*
	2	0.300/(1)*	0.300/(1)*	0.300/(1)*

Table 4: Optimal allocation for $(\kappa, \beta) = (15, 0.3)$

Distribution	π_0	0.679		
	π_1	0.162		
	π_2	0.054		
	π_3	0.105		
Transfer	τ_0	0		
	τ_1	0		
	τ_2	0.668		
production and payment in (k, k') meeting	$k \backslash k'$	1	2	3
	0	0.207/(1)*	0.305/(2)*	0.305/(2)*
	1	0.098/(1)*	0.098/(1)*	0.106/(2)*
	2	0/(0)*	0.009/(1)*	0.009/(1)*

across two upper bounds, optimal money supply is less volatile with the higher upper bound. This is intuitive as the wealth level can be more finely recorded with the richer money holdings. It does not exceed a half of B in any cases.

Table 5: Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 2$

$\kappa \backslash \beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0.0285	0.1145	0.1810	0.2330	0.2765	0.3190	0.3950	0.4305	0.4505	0.4655
15	-	0.0390	0.1120	0.1715	0.2225	0.2660	0.3445	0.4130	0.4530	0.4675
12	-	-	0.0510	0.1160	0.1715	0.2200	0.3065	0.3815	0.4485	0.4660
10	-	-	-	0.0645	0.1240	0.1765	0.2700	0.3540	0.4290	0.4615
8	-	-	-	-	0.0595	0.1160	0.2185	0.3125	0.4005	0.4565
6	-	-	-	-	-	0.0275	0.1365	0.2440	0.3515	0.4495
5	-	-	-	-	-	-	0.0770	0.1895	0.3065	0.4300
4	-	-	-	-	-	-	-	0.1125	0.2400	0.3825
3	-	-	-	-	-	-	-	-	0.127	0.290
2	-	-	-	-	-	-	-	-	-	0.0835

Table 6: Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 3$

$\kappa \backslash \beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0.029	0.138	0.2063	0.2547	0.258	0.281	0.3187	0.3407	0.3837	0.4437
15	-	0.041	0.133	0.195	0.2433	0.2413	0.2923	0.335	0.3933	0.436
12	-	-	0.054	0.136	0.1937	0.194	0.27	0.329	0.3837	0.4293
10	-	-	-	0.07	0.143	0.197	0.2423	0.3243	0.3607	0.4227
8	-	-	-	-	0.064	0.1333	0.194	0.282	0.336	0.4123
6	-	-	-	-	-	0.031	0.154	0.2147	0.302	0.392
5	-	-	-	-	-	-	0.087	0.1653	0.2753	0.362
4	-	-	-	-	-	-	-	0.1283	0.2077	0.3167
3	-	-	-	-	-	-	-	-	0.147	0.2557
2	-	-	-	-	-	-	-	-	-	0.101

The lottery is potentially useful as money is indivisible and money holding is limited (due to the concavity of utility function, it is not helpful to use lottery for production level). We computed the percentage of trade meetings where the lottery is used to any degree to find how frequently the lottery is used. It tends to be used more with high discount rate and risk aversion, but is not monotonically increasing. As the value of money at optimum increases with the discount rate, the lottery gets used more often to overcome the indivisibility.

We construct a similar measure for the trade meetings where consumers are making take-it-or-leave-it offer. As this trading mechanism is popular for its simplicity, it is interesting to see whether such trading mechanism is close to the optimal one. Although our measure does not capture the welfare loss from imposing take-it-or-leave-it trading mechanism, it is indicative. In one example, almost 50% of meetings are departing from take-it-or-leave-it trading mechanism. Again, this is observed when both parameters are high. Considering the optimum with $B = 1$ sheds some light to the reason why the departing if salient with high discount rate. When the upper bound is one and discount rate is sufficiently high, the optimum is achievable by making the half of population holding money and producers producing the first-best level of production. Consumers don't get all the surplus in that case, as it will make producers to produce too much. The result here shows the similar pattern, that is, consumers don't get all the surplus when discount rate is high.

References

Alexei Deviatov. Money creation in a random matching model. *Topics in Macroeconomics*, 6(3), 2006.

Timothy J Kehoe, David K Levine, and Michael Woodford. The optimum quantity of money revisited. *Economic analysis of markets and games: Essays in honor of Frank Hahn*, pages 501–526, 1992.

David K Levine. Asset trading mechanisms and expansionary policy. *Journal of Economic Theory*, 54(1):148–164, 1991.

A Mas-Collel, Michael D Whinston, and Jerry R Green. Micreconomic theory, 1995.

Shouyong Shi. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67(2):467–496, 1995.

Alberto Trejos and Randall Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, pages 118–141, 1995.

Neil Wallace. Optimal money creation in "pure currency" economies: a conjecture. *The Quarterly Journal of Economics*, 129(1):259–274, 2014.

Tao Zhu. Existence of a monetary steady state in a matching model: indivisible money. *Journal of Economic Theory*, 112(2):307–324, 2003.

A Pairwise Core Constraint

The problem (8) can be rewritten as

$$\begin{aligned}
&\max_{\lambda_c} u\left(\beta \sum_{0\le i\le k+k'} \lambda_c(i) w\left(k+k'-i\right) - \beta w(k) - \vartheta(k,k')\right) + \beta \sum_{0\le i\le k+k'} \lambda_c(i) w(i) \\
&\text{s.t. } [\mu_q^{k,k'}] - \left(\beta \sum_{0\le i\le k+k'} \lambda_c(i) w\left(k+k'-i\right) - \beta w(k) - \vartheta(k,k')\right) \le 0 \\
&\quad [\mu_0^{k,k'}(i)] - \lambda_c(i) \le 0 \text{ for all } i \\
&\quad [\mu_{\text{sum}}^{k,k'}] 1 - \sum_{0\le i\le k+k'} \lambda_c(i) = 0
\end{aligned}$$

where μ's (the variables in square brackets before the constraints) denote multipliers. Since all the constraints are linear, the constraint qualification is satisfied. As the objective is concave, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for the optimality. The KKT condition can be stated as follows:

$$0 = \left[u'\left(q(k,k')\right) + \mu_q^{k,k'}\right] \beta w\left(k+k'-i\right) + \beta w\left(i\right) + \mu_0^{k,k'}(i) + \mu_{\text{sum}}^{k,k'} \text{ for all } i, \tag{9}$$

$$0 \ge -q(k,k'), \tag{10}$$

$$0 \ge -\lambda_c^{k,k'}(i) \text{ for all } i, \tag{11}$$

$$0 = 1 - \sum_{0\le i\le k+k'} \lambda_c^{k,k'}(i), \tag{12}$$

$$0 = \mu_q^{k,k'} q(k,k'), \tag{13}$$

$$0 = \mu_0^{k,k'}(i) \lambda_c^{k,k'}(i) \text{ for all } i, \tag{14}$$

and

$$q(k,k') = \beta \sum_{0\le i\le k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) - \beta w(k) - \vartheta(k,k').$$

In the following, we show that the last condition and the surplus variable $\vartheta(k,k')$ are redundant.

The original planner's problem (Problem O), which is with ϑ, can be stated as

$$\begin{array}{lll}\max\limits_{\vartheta,q,\lambda,\delta,\tau,\pi,v,w} & & \dfrac{\sum_{0\le k\le B}\sum_{0\le k'\le B}\pi_k\pi_{k'}\left[u\left(q(k,k')\right)-q(k,k')\right]}{K\left(1-\beta\right)} \\ \text{s.t.} & \{\text{IRs}\} & -q(k,k')+\beta\sum_{0\le i\le k+k'}\lambda_c^{k,k'}(i)w(k+k'-i)\ge\beta w(k) \\ & & (\text{Other IRs}) \\ & & \vartheta(k,k')\ge 0 \\ & \{\text{PCs}\} & -q(k,k')+\beta\sum_{0\le i\le k+k'}\lambda_c^{k,k'}(i)w(k+k'-i)=\beta w(k)+\vartheta(k,k') \\ & & (\text{Other PCs}) \\ & & (\text{All the other constraints})\end{array}$$

Note that ϑ appears only in the pairwise core constraints explicitly written above. The reduced problem (Problem R) is

$$\begin{array}{lll}\max\limits_{q,\lambda,\delta,\tau,\pi,v,w} & & \dfrac{\sum_{0\le k\le B}\sum_{0\le k'\le B}\pi_k\pi_{k'}\left[u\left(q(k,k')\right)-q(k,k')\right]}{K\left(1-\beta\right)} \\ \text{s.t.} & \{\text{IRs}\} & -q(k,k')+\beta\sum_{0\le i\le k+k'}\lambda_c^{k,k'}(i)w(k+k'-i)\ge\beta w(k) \\ & & (\text{Other IRs}) \\ & \{\text{PCs}\} & (\text{Other PCs}) \\ & & (\text{All the other constraints})\end{array}$$

Let S_O and S_R denote the set of solutions for Problem O and that for Problem R, respectively. Define $\tilde{S}_O = \{(q,\lambda,\delta,\tau,\pi,v,w)|\exists\vartheta,(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)\in S_O\}$. The precise claim we want to show is the following.

Claim 1. $\tilde{S}_O = S_R$.

Proof. We prove two inclusion relationships, $\tilde{S}_O \subset S_R$ and $S_R \subset \tilde{S}_O$.

- ($\tilde{S}_O \subset S_R$) Let $(q,\lambda,\delta,\tau,\pi,v,w)\in\tilde{S}_O$. By definition, there exists ϑ such that $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)\in S_O$. The feasibility of $(\vartheta,q,\lambda,\delta,\tau,\pi,v,w)$ in Problem O clearly implies the feasibility of $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R. Suppose, by way of contradiction, $(q,\lambda,\delta,\tau,\pi,v,w)\notin S_R$: there exists $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ such that $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ is feasible and better than $(q,\lambda,\delta,\tau,\pi,v,w)$ in Problem R. Define $\tilde{\vartheta}$, using

  $$\tilde{q}(k,k') \;=\; \beta\sum_{0\le i\le k+k'}\tilde{\lambda}_c^{k,k'}(i)\tilde{w}(k+k'-i)-\beta\tilde{w}(k)-\tilde{\vartheta}(k,k').$$

  Then, the IR implies $\tilde{\vartheta}(k,k')\ge 0$, so $(\tilde{\vartheta},\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ satisfies all the constraints in Problem O. Also, the supposition that $(\tilde{q},\tilde{\lambda},\tilde{\delta},\tilde{\tau},\tilde{\pi},\tilde{v},\tilde{w})$ is strictly better than

$(q, \lambda, \delta, \tau, \pi, v, w)$ in Problem R implies that $(\tilde{\vartheta}, \tilde{q}, \tilde{\lambda}, \tilde{\delta}, \tilde{\tau}, \tilde{\pi}, \tilde{v}, \tilde{w})$ is strictly better than $(q, \lambda, \delta, \tau, \pi, v, w)$ in Problem O. That contradicts the assumption $(\vartheta, q, \lambda, \delta, \tau, \pi, v, w) \in S_O$, so we have $(q, \lambda, \delta, \tau, \pi, v, w) \in S_R$.

- $(S_R \subset \tilde{S}_O)$ Let $(q, \lambda, \delta, \tau, \pi, v, w) \in S_R$. Define ϑ, using

$$q(k, k') = \beta \sum_{0 \leq i \leq k+k'} \lambda_c^{k,k'}(i) w(k + k' - i) - \beta w(k) - \vartheta(k, k').$$

Then, the IR implies $\vartheta(k, k') \geq 0$, so $(\vartheta, q, \lambda, \delta, \tau, \pi, v, w)$ is feasible in Problem O. The optimality of $(q, \lambda, \delta, \tau, \pi, v, w)$ in Problem R implies the optimality of $(\vartheta, q, \lambda, \delta, \tau, \pi, v, w)$ in Problem O in the way similar to the above. So we have $(q, \lambda, \delta, \tau, \pi, v, w) \in \tilde{S}_R$.

□